Let us switch to a new topic:

- **Graphs**

Introduction to Graphs

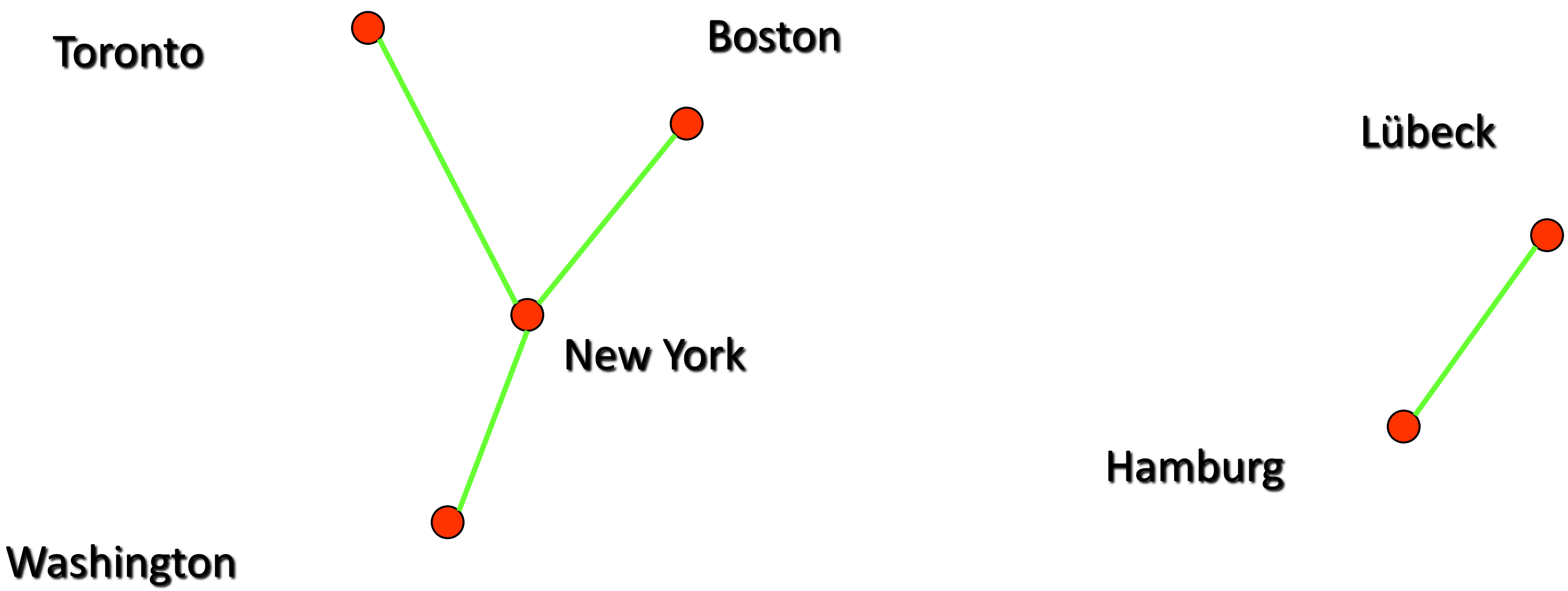
- **Definition:** A **simple graph** $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of **unordered pairs** of distinct elements of V called edges.
- For each $e \in E$, $e = \{u, v\}$ where $u, v \in V$.
- An undirected graph (not simple) may contain loops. An edge e is a loop if $e = \{u, u\}$ for some $u \in V$.

Introduction to Graphs

- **Definition:** A **directed graph** $G = (V, E)$ consists of a set V of vertices and a set E of edges that are ordered pairs of elements in V.
- For each $e \in E$, $e = (u, v)$ where $u, v \in V$.
- An edge e is a loop if $e = (u, u)$ for some $u \in V$.
- A simple graph is just like a directed graph, but with no specified direction of its edges.

Graph Models

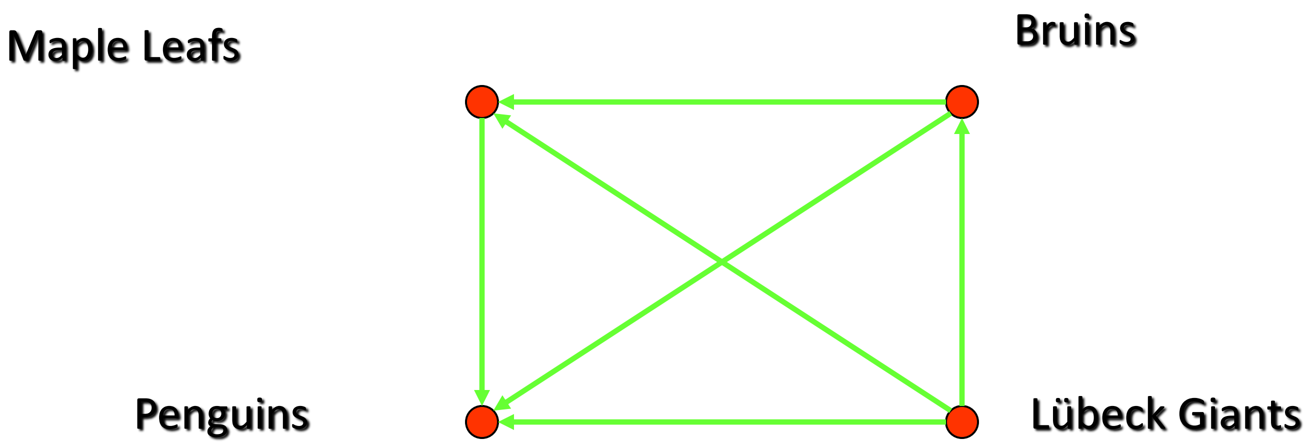
- **Example I:** How can we represent a network of (bi-directional) railways connecting a set of cities?
- We should use a **simple graph** with an edge {a, b} indicating a direct train connection between cities a and b.



Graph Models

- **Example II:** In a round-robin tournament, each team plays against each other team exactly once. How can we represent the results of the tournament (which team beats which other team)?

- We should use a **directed graph** with an edge (a, b) indicating that team a beats team b.



Graph Terminology

- **Definition:** Two vertices u and v in an undirected graph G are called **adjacent** (or **neighbors**) in G if {u, v} is an edge in G.
- If e = {u, v}, the edge e is called **incident with** the vertices u and v. The edge e is also said to **connect** u and v.
- The vertices u and v are called **endpoints** of the edge {u, v}.

Graph Terminology

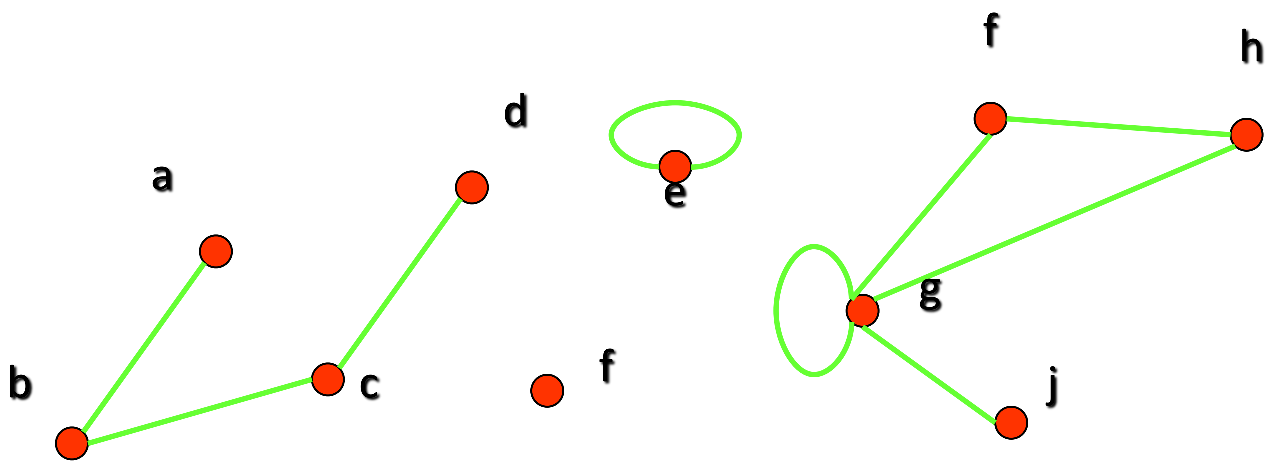
- **Definition:** The **degree** of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.
- In other words, you can determine the degree of a vertex in a displayed graph by **counting the lines** that touch it.
- The degree of the vertex v is denoted by **deg(v).**

Graph Terminology

- A vertex of degree 0 is called **isolated**, since it is not adjacent to any vertex.
- **Note:** A vertex with a **loop** at it has at least degree 2 and, by definition, is **not isolated**, even if it is not adjacent to any **other** vertex.
- A vertex of degree 1 is called **pendant**. It is adjacent to exactly one other vertex.

Graph Terminology

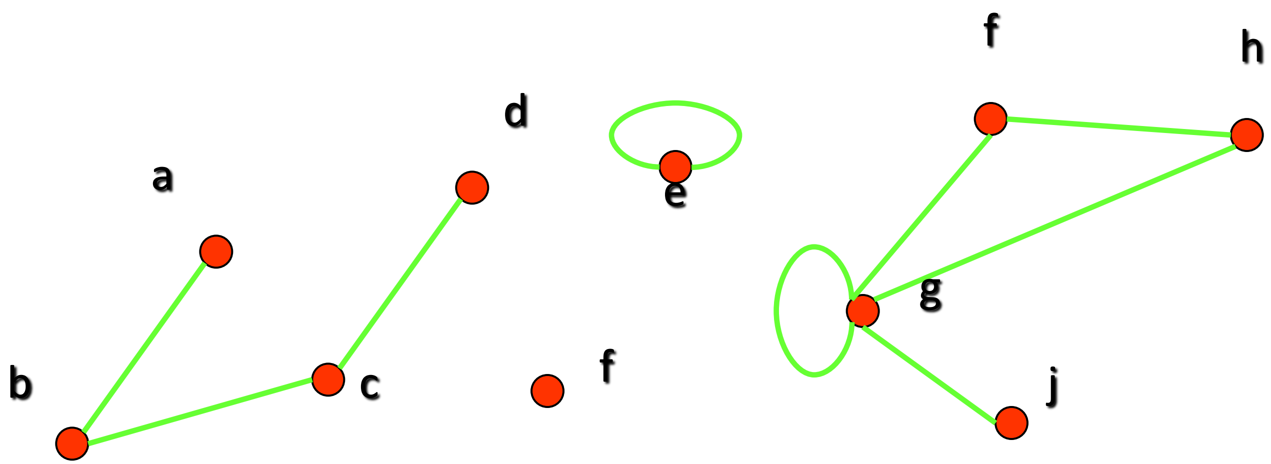
- **Example:** Which vertices in the following graph are isolated, which are pendant, and what is the maximum degree? What type of graph is it?



Solution: Vertex f is isolated, and vertices a, d and j are pendant. The maximum degree is deg(g) = 5.
This graph is a pseudograph (undirected, loops).

Graph Terminology

- Let us look at the same graph again and determine the number of its edges and the sum of the degrees of all its vertices:



Result: There are 9 edges, and the sum of all degrees is 18. This is easy to explain: Each new edge increases the sum of degrees by exactly two.

Graph Terminology

- **The Handshaking Theorem:** Let $G = (V, E)$ be an undirected graph with e edges. Then
- $2e = \sum_{v \in V} \deg(v)$

- **Example:** How many edges are there in a graph with 10 vertices, each of degree 6?
- **Solution:** The sum of the degrees of the vertices is $6 \cdot 10 = 60$. According to the Handshaking Theorem, it follows that $2e = 60$, so there are 30 edges.

Graph Terminology

- **Theorem:** An undirected graph has an even number of vertices of odd degree.
- **Proof:** Let V1 and V2 be the set of vertices of even and odd degrees, respectively (Thus $V1 \cap V2 = \emptyset$, and $V1 \cup V2 = V$).
- Then by Handshaking theorem
    - $2|E| = \sum_{v \in V} \deg(v) = \sum_{v \in V1} \deg(v) + \sum_{v \in V2} \deg(v)$
- Since both $2|E|$ and $\sum_{v \in V1} \deg(v)$ are even,
- $\sum_{v \in V2} \deg(v)$ must be even.
- Since deg(v) if odd for all $v \in V2$, $|V2|$ must be even.
- **QED**

Graph Terminology

- **Definition:** When (u, v) is an edge of the graph G with directed edges, u is said to be **adjacent to** v, and v is said to be **adjacent from** u.
- The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal vertex** of (u, v).
- The initial vertex and terminal vertex of a loop are the same.

Graph Terminology

- **Definition:** In a graph with directed edges, the **in-degree** of a vertex v, denoted by **$deg^-(v)$**, is the number of edges with v as their **terminal vertex**.
- The **out-degree** of v, denoted by **$deg^+(v)$**, is the number of edges with v as their initial vertex.

- **Question:** How does adding a loop to a vertex change the in-degree and out-degree of that vertex?
- **Answer:** It increases both the in-degree and the out-degree by one.

Graph Terminology

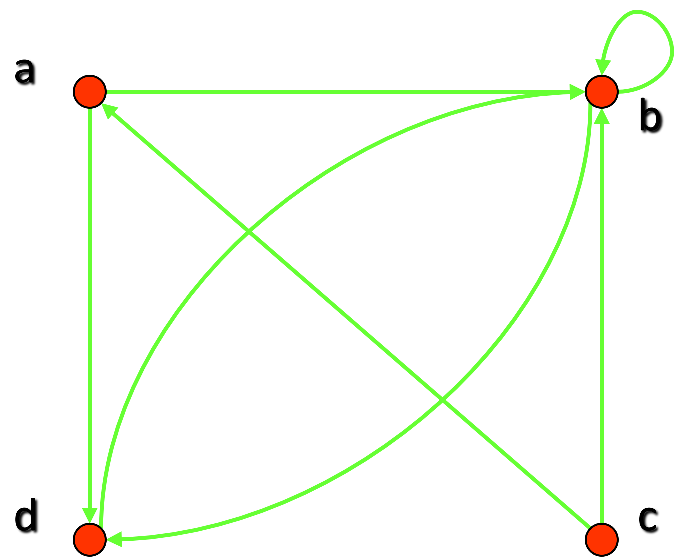
•**Example:** What are the in-degrees and out-degrees of the vertices a, b, c, d in this graph:

$\deg^-(a) = 1$

$\deg^+(a) = 2$

$\deg^-(b) = 4$

$\deg^+(b) = 2$



$\deg^-(d) = 2$

$\deg^+(d) = 1$

$\deg^-(c) = 0$

$\deg^+(c) = 2$